

Improved Preservation Technology for Non-Instantaneous Deteriorating Inventory Using Boundary Condition Estimation

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Abstract: Various forms of preservation technology nowadays allow businesses to handle valuable perishable items with greater flexibility. Even with a wide variety of preservation techniques, the mathematical modelling of its implementation in EOQ literature remains rigid. The paper aims to integrate an improved preservation technology in a non-instantaneous deteriorating inventory model for businesses maximizing their average total cycle profit. The improved preservation technology furthers the delay to the time within the cycle where deterioration begins and enhances the durability of inventory that allows operators to employ a less prudent holding facility. Another improvement in this area is the accurate accumulation of preservation cost depending on the inventory level at hand. The conventional EOQ method of forming the objective function before choosing the optimal values for our two decision variables (Cycle time and level of preservation) is undertaken. The cycle time is split in two, differing in their inventory process (deterioration beginning in the second period). The time when deterioration begins is derived using the model's boundary conditions, a first attempt within the area. The optimal solution set is solved for a numerical example using an algorithm to demonstrate the model and prove the global nature of the solution. An investigation into the gains from the improved preservation technology is conducted by dissecting the effects within each individual component within the objective function. 3 separate channels by which this improved preservation technology modelling benefits the business model is found namely shifting to the higher profitable period, effects towards preservation affected costs and the returns to scale from successively increasing preservation levels. Sensitivity analysis is conducted to demonstrate and confirm the findings. The paper discovers great benefits from such an improved modelling that warrants further attention within the scope of preserved inventory models, especially on how levels of preservation could influence the traditional decision variable optimized such as cycle time or ordering frequency. Findings of the paper would have significant benefits to different inventory models with its own delay before deterioration and holding facility requirement.

Keywords: Operational Research, Mathematical Modelling, Inventory, Preservation, Non-Instantaneous

1. Introduction

Deterioration within inventory occurs in various industries such as imaging, health sector and perishable food-items. Due to its universal relevance, companies employ preservation technologies with varying intensity to control the deterioration rate and lower losses owing to inventory loss via deterioration. Traditionally, preservation technology has been treated merely as a machinery installment incurring

constant cost per time during the operation cycle [10, 18, 21, 36]. This approach has been adopted by nearly all papers that involves a utilization of preservation technology [12, 13, 20, 28, 35]. While this approach is straightforward to model as merely a multiple of the total cycle time, a shortcoming from this approach is the assumed constant preservation cost independent of the inventory level at hand.

Realistically, this treatment neglects the varying inventory levels held during the cycle that would require varying

amounts of preservation. For example, one cannot expect a single air conditioning unit to effectively preserve 1 tonne of perishable fruits as well as it preserves 10 kg of the same item. Therefore a more realistic preservation technology would incur costs depending on the amount of time that it is being employed and also depending on the amount of inventory it preserves. An improvement from the previous assumption would enable inventory operators the flexibility to incur preservation costs according to the size of inventory at hand for each time period. This type of Inventory Proportional Preservation Cost has never been adopted in any previous paper.

Within the scope of deteriorating inventory models, literature can be categorized by the start of the deterioration process within the cycle. Earlier papers that did not consider preservation adopted the straightforward approach where deterioration begins at the start of the cycle [1, 3, 8, 9, 19, 23, 31, 32, 33]. This approach has been adopted by recent studies that now consider preservation to affect the deterioration process [4, 6, 7, 12, 20, 22, 29]. These types of deterioration

process are commonly known as Instantaneously Deteriorating inventory. This might apply to some very delicate and fragile inventory that begins to deteriorate the moment it is stored such as radioactive isotopes or oil paintings. Currently however, this type of deterioration is rare since majority of perishable inventory would take a while before it starts being spoilt fully beyond consumption quality. As an alternative, a delay before deterioration begins affecting the inventory process during the cycle time is considered. For example, deterioration could begin as inventory is transferred from the factory to the warehouse [2, 5]. These types of inventory are classified as Non-Instantaneously Deteriorating Inventory models [11, 14, 25, 34]. Although this is an improved and more realistic assumption, majority of papers do not elaborate on how the delay is determined [13, 15, 16, 26, 28]. Usually the delay before deterioration begins is taken as an exogenously determined factor not affected by the decision variables as summarized in table 1 below:

Table 1. Approach for Start of Deterioration in Recent Literature.

Paper	Instantaneous	Exogenously Determined Delay	Endogenously Controllable Delay
Tao et. al (2019)	✓		
Dye and Yang (2017)	✓		
Mishra (2014)		✓	
Yang et. al (2018)		✓	
Zhang et. al (2015)		✓	
Singh, Mishra and Pattanayak (2017)		✓	
Mishra, Wu, Tsao and Tseng (2019)		✓	
Maihami and Karimi (2014)	✓		
Li, He, Zhou and Wu (2019)			✓
Singh and Sharma (2013)	✓		
Huang, He and Li (2018)	✓		

With various preservation technologies such as refrigeration, pesticide and handling instruments, it is surprising that preservation has not been seriously considered as a decision variable that could affect the delay before preservation [17, 24, 25, 29, 38]. Up to date only one paper addressed the delay period within their model, where the delay is affected by the preservation level though it was not explicitly derived and described [30]. Suggestively, a higher preservation level would further delay the time before deterioration process begins within the time cycle. This would occur since higher preservation levels would decrease the perishability of inventories.

Finally, the implementation of preservation technologies could also give positive externalities within the Model that benefits the operators. These benefits could come with the form of reducing related costs such as emission penalties or reduced inventory holding costs as found commonly in inventory models with green initiatives to conserve the environment [15, 27, 29, 34]. These positive benefits could also come from tax leeway given by certain countries for operators that adopt preservation technologies. In other cases, perishable products that become more durable with preservation could be held in less delicate and expensive storage facilities, lowering the inventory holding cost.

The paper starts by deriving the estimate equations for the initial period where no deterioration is in effect and the second period where deterioration is present. Using the estimate equations from both periods, the point within the cycle where deterioration begins (t_0) is derived. This point within the cycle time (t_0) is further explored through its expression. Following on, the objective function is constructed to be maximised. The order conditions for optimality are then derived for the two decision variable which is the length of cycle for the second period (T) and also the level preservation (β). We then solve a numerical example to find the solution for a given scenario using a simple algorithm. The optimal decision variable values are then verified as the global solution using the Hessian matrix. To thoroughly investigate the advantages of employing preservation technology in deteriorating inventories, the gains in profit from implementing this improved preservation technology are deconstructed. The individual terms are examined to specify by which channels preservation technology may lead to an increase in profit or avoid loss. Next, some sensitivity analysis is conducted to verify the outcomes found. Finally, improvements to be considered in future research are suggested before concluding remarks are made on the outcomes from this research.

2. Notations and Assumptions

2.1. Notations

Table 2. Description of notations used in the paper.

Notation	Description
$I_1(t), I_2(t)$	Inventory level for each period
t	Current time within each period
t_0	First period cycle time
T	Second period cycle time
$[0, t_0]$	First period cycle range
$[0, T]$	Second period cycle range
Q	Inventory level at start
α	Initial deterioration rate
β	Preservation level
$\theta = \frac{\alpha}{1+\beta}$	Effective deterioration rate
λ	Inventory proportional demand rate
D	Constant demand rate in period 2
OC	Ordering cost/unit
DC	Deterioration cost/unit
$HC(\beta)$	Holding cost/unit/time
$PC(\beta)$	Preservation cost/unit/time
$I_1^{T(\beta)}$	Total inventory held in first period with preservation
$I_2^{T(\beta)}$	Total inventory held in second period with preservation

2.2. Assumption

There are now two separate periods ($0 < t_1 < t_0$) and ($0 < t_2 < T$) with two different inventory processes. Deterioration takes into effect from the end of the first period (t_0) or the start of the second period, where the inventory process changes from Eq. (1) to Eq. (3).

Preservation technology enters as a discounting factor as shown in (4) that reduces the proportion of deterioration, with diminishing marginal returns. No shortages are allowed during both cycle periods [$I_1(t), I_2(t) > 0$]. Total demanded and deteriorated inventory is less than the amount of inventory available at any point in the cycle time [$(I(t)*(\lambda + \alpha)) + D < I$].

Ordering Cost per unit applies to each unit ordered initially at the start of the cycle ($OC=C_o$). Deterioration Cost per unit equals revenue lost had the inventory been sold ($DC=C_d=P$). Preservation Cost per time-inventory depends on the level of preservation employed, [$PC(\beta)=(C_p)^*(\beta)$] when [$\beta>0$], and [$PC=0$] when [$\beta=0$]. Preservation investment discounts per time-inventory holding cost depending on the level of preservation when employed, [$HC(\beta)= (C_h)/(\beta)$] when [$\beta>0$], and [$HC(\beta) = C_h$] when [$\beta=0$].

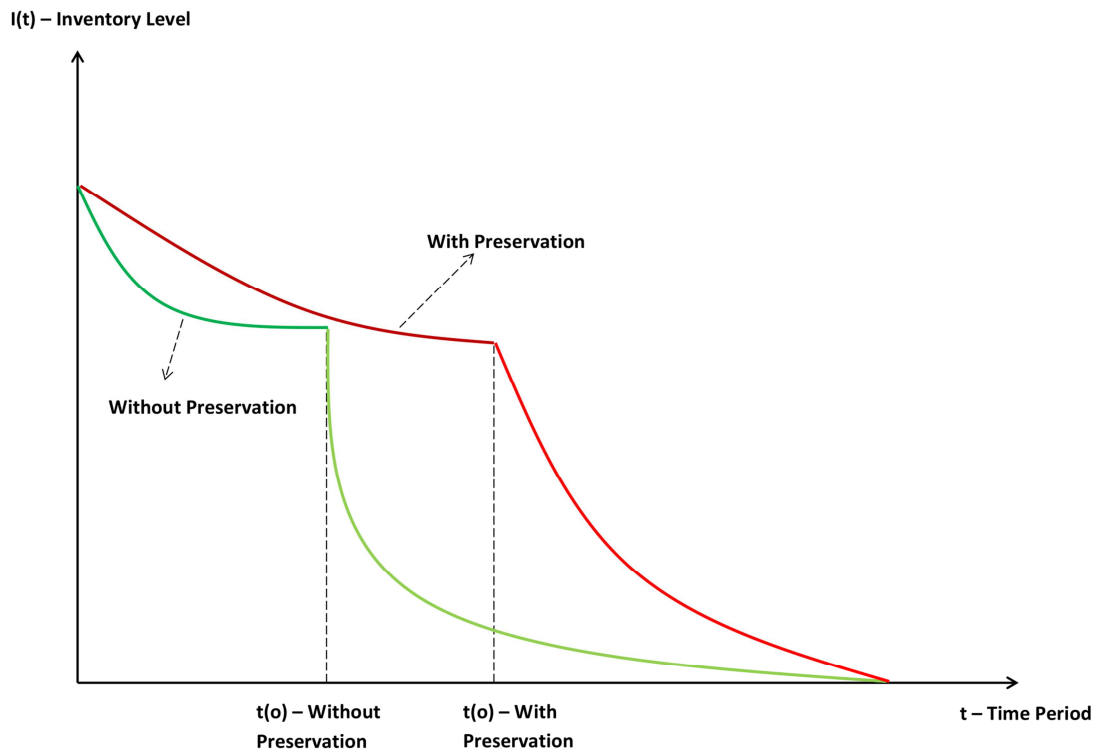


Figure 1. Inventory Process With and Without Preservation Technology Implementation.

3. Inventory Process

3.1. Period 1 ($0 < t_1 < t_0$)

In the first period, there are no deterioration and the demand function consists of only a proportional demand (λ) dependent on the current inventory level at hand:

$$\frac{dI_1(t)}{dt} = -\lambda I_1(t) \quad (1)$$

Solving the differential equation above with the boundary condition of ($I=Q$) at ($t_1=0$) gives estimation for Inventory level in the first period:

$$I_1(t) = \frac{Q}{\exp(\lambda t)} \quad (2)$$

3.2. Period 1 ($0 < t_2 < T$)

In the second period, effective deterioration (θ) is now in effect as a proportion of current inventory level at hand

$(\theta I_1(t))$ while demand now includes a small constant amount (D) to ensure there is still demand even with minimal levels of inventory:

$$\frac{dI_2(t)}{dt} = -D - \lambda I_1(t) - \theta I_1(t) \quad (3)$$

Depending on the level of preservation technology used, effective deterioration is reduced at a diminishing marginal rate (Lemma 1). We can check this property by the partial differential of the effective deterioration rate with respect to the preservation level.

$$\theta(\beta) = \frac{\alpha}{1+\beta} \quad (4)$$

$$\frac{\partial \theta}{\partial \beta} = \frac{-\alpha}{(1+\beta)^2}, \quad \frac{\partial^2 \theta}{\partial \beta^2} = \frac{2\alpha}{(1+\beta)^3} \quad (5)$$

Solving the differential equation above with the boundary condition of $(I(T)=0)$ at $(t_2=T)$ gives the estimation for Inventory level during the second period:

$$I_2(t) = \frac{D}{\lambda+\theta} [\exp((\lambda+\theta)(T-t)) - 1] \quad (6)$$

3.3. Start of Deterioration (t_0)

Equating $I_1(t_0)$ with $I_2(0)$ as per our assumption on the two time periods gives out the expression of (t_0) that ensures the continuity and consistency from the first period to the second period:

$$I_1(t_0) = I_2(0) \text{ or } \frac{Q}{\exp(\lambda t_0)} = \frac{D}{\lambda+\theta} [\exp((\lambda+\theta)T) - 1] \quad (7)$$

$$t_0 = \frac{1}{\lambda} \ln \left[\frac{Q(\lambda+\theta)}{[\exp((\lambda+\theta)T) - 1]D} \right]$$

This expression gives the point within the cycle time where deterioration kicks in or the length of the first period within our cycle. Thus after (t_0) , the inventory process and estimate changes from Eq. (2) to Eq. (6). As seen within the expression, (t_0) depends on the length of second period cycle time (T) and preservation (β) . Checking the first order condition lets us check how the decision variables affect (t_0) :

$$\frac{\partial t_0}{\partial T} = \frac{1}{\lambda} \left[\frac{(\lambda+\theta) \exp((\lambda+\theta)T)}{[1 - \exp((\lambda+\theta)T)]} \right] \quad (8)$$

$$\frac{\partial t_0}{\partial \beta} = \frac{\alpha}{\lambda(1+\beta)^2} \left\{ \frac{T \exp((\lambda+\theta)T)}{[\exp((\lambda+\theta)T) - 1]} - \frac{(\lambda+\theta)}{1} \right\} \quad (9)$$

We can prove that employing a higher preservation level reduces effective deterioration rate (θ) which prolongs the delay (length of first period) before deterioration kicks-in (Lemma 2):

$$t_0^{(\beta>0)} > t_0^{(\beta=0)} \quad (10)$$

when $t_0^{(\beta>0)} = \frac{1}{\lambda} \ln \left[\frac{Q(\lambda+\theta)}{[\exp((\lambda+\theta)T) - 1]D} \right]$, $\theta(\beta) = \frac{\alpha}{1+\beta}$

Proof: The statement to be proven is, t_0 increases as θ decreases or as β increases.

We impose the condition for $(t_0 > 0)$ requiring the terms in the early bracket to be positive. We observe the behavior of

the term within the curly bracket as (β) and thus (θ) changes.

The numerator term of $Q(\lambda+\theta)$ rises linearly while the term in the denominator $[\exp((\lambda+\theta)T) - 1]D$ rises exponentially as (θ) increases. Therefore the term in the curly bracket and thus (t_0) decreases as (θ) increases. This would occur when (β) decreases.

Conversely this would prove our statement and Lemma 2 to be proven.

3.4. Inventory Held During Cycle

The total inventory held during the first period could be obtained by integrating the first period inventory estimate over the time cycle of the first period:

$$\int_0^{t_0} I_1(t) dt = \frac{Q}{\lambda} \left[1 - \frac{1}{\exp(\lambda t_0)} \right] = I_1^{T(\beta)} \quad (11)$$

It can be proven that inventory held in the first period is greater with (higher) use of preservation technology (Lemma 3).

Proof: From the expression above, an alternate first period inventory held with no (lower) preservation is as below:

$$I_1^{T(\beta)} = \int_0^{t_0} I_1(t)^{(\beta>0)}(t) dt = \frac{Q}{\lambda} \left[1 - \frac{1}{\exp(\lambda t_0(\beta>0))} \right],$$

$$I_1^{T(0)} = \int_0^{t_0} I_1(t)^{(\beta=0)}(t) dt = \frac{Q}{\lambda} \left[1 - \frac{1}{\exp(\lambda t_0(\beta=0))} \right]$$

Since via Lemma 2:

$$t_0^{(\beta>0)} > t_0^{(\beta=0)}$$

It can be clearly seen:

$$\frac{Q}{\lambda} \left[1 - \frac{1}{\exp(\lambda t_0(\beta>0))} \right] > \frac{Q}{\lambda} \left[1 - \frac{1}{\exp(\lambda t_0(\beta=0))} \right]. \quad \square$$

The total inventory held during the second period could be obtained by integrating the inventory estimate over the time cycle of the second period:

$$\int_0^T I_2(t) dt = \frac{D}{\lambda+\theta} \left[\frac{\exp((\lambda+\theta)T) - 1}{\lambda+\theta} - T \right] = I_2^{T(\beta)} \quad (12)$$

We now prove that inventory held in the second period is smaller with (higher) use of preservation technology (Lemma 4):

Proof: Similarly as the first period. The total inventory held during the second period with and without (lower) preservation is given by:

$$I_2^{T(\beta)} = \int_0^T I_2(t)^{(\beta>0)} dt = \frac{D}{\lambda+\theta} \left[\frac{\exp((\lambda+\theta)T) - 1}{\lambda+\theta} - T \right],$$

$$I_2^{T(0)} = \int_0^T I_2(t)^{(\beta=0)} dt = \frac{D}{\lambda+\alpha} \left[\frac{\exp((\lambda+\alpha)T) - 1}{\lambda+\alpha} - T \right]$$

We estimate the term in the early bracket using a Taylor Series expansion:

$$\int_0^T I_2(t)^{(\beta>0)} dt = \frac{D}{\lambda+\theta} \left[\frac{\left\{ (\lambda+\theta) + \frac{(\lambda+\theta)^2}{2} \right\}}{\lambda+\theta} - T \right],$$

$$\int_0^T I_2(t)^{(\beta=0)} dt = \frac{D}{\lambda+\alpha} \left[\frac{\{(\lambda+\alpha) + \frac{(\lambda+\alpha)^2}{2}\}}{\lambda+\alpha} - T \right]$$

Since:

$$\text{Exp}[(\gamma)(T)] \approx 1 + [(\gamma)(T)] + \frac{[(\gamma)(T)]^2}{2} + \frac{[(\gamma)(T)]^3}{6} + \dots$$

The Lemma simplifies into proving:

$$D \left[\frac{T^2}{2} + \frac{(\lambda+\theta)T^3}{6} \right] < D \left[\frac{T^2}{2} + \frac{(\lambda+\alpha)T^3}{6} \right]$$

Which holds true because through Lemma 1 ($\alpha > \theta$). \square

4. Objective Function

With all the important equations obtained, the objective function can be constructed from the Total Revenue (TR) and Total Cost (TC).

4.1. Total Revenue

Total demand includes the small constant demand (D) over the second period (T) and the proportional demand (λ) dependent on the inventory level held during both periods. Total revenue is thus the total demand multiplied by the selling price (P):

$$TR = PD[T] + P\lambda \int_0^{t_0} I_1(t) dt + P\lambda \int_0^T I_2(t) dt \quad (13)$$

4.2. Total Cost

Total cost includes Ordering cost (OC), Holding cost (HC), Preservation cost (PC) and Deterioration cost (DC). While ordering cost is just a multiple of the initial order quantity (Q), the two other costs are inventory dependent and is a unit/time type of cost. Additionally they depend on the level of preservation employed (β). Finally, Deterioration cost depends on the proportion of inventory that deterioration and only occurs during the second period.

$$TC = C_o Q + \left[\frac{C_h}{\beta} + C_p \beta \right] \int_0^{t_0} I_1(t) dt + \left[\frac{C_h}{\beta} + C_p \beta + \frac{C_d \alpha}{1+\beta} \right] \int_0^T I_2(t) dt \quad (14)$$

4.3. Profit Function

The profit function can thus be constructed using the total revenue and total cost functions. Total inventory held during period 1 and period 2 are common terms found in both functions allowing the simplification of the profit function:

$$\pi = TR - TC = [PDT] + [P\lambda - \frac{C_h}{\beta} - C_p \beta] \int_0^{t_0} I_1(t) dt + [P\lambda - \frac{C_h}{\beta} - C_p \beta - \frac{C_d \alpha}{1+\beta}] \int_0^T I_2(t) dt - C_o Q \quad (15)$$

The finalized objective function is the average total cycle profit ($ATC\pi$) optimized by choosing the preservation factor (β) and the second period cycle time (T). Notice the first period cycle time (t_0) would also depend on the two decision

variables (β) and (T) through its expression:

$$\max_{(\beta, T)}: [ATC\pi] = \pi / T^{T(\beta)}$$

$$\text{Where } T^{T(\beta)} = t_0 + T = \left\{ \frac{1}{\lambda} \ln \left[\frac{Q(\lambda+\theta)}{\exp((\lambda+\theta)T-1)b} \right] \right\} + T$$

4.4. Order Conditions

Since the two decision variables (β) and (T) exists within multiple expressions in the objective function, it is easier to organize the order conditions with these expressions. The full expressions and their derivatives with respect to the two decision variables are provided in the appendix. The first and second order condition for the average profit function over the whole cycle time ($\frac{\pi}{t_0+T}$) with (β) and (T) as decision variables are given as:

$$ATC\pi(\beta, T) = \frac{\pi}{T^{T(\beta)}}$$

$$\frac{\partial ATC\pi}{\partial T} = \frac{\pi'_T}{T^{T(\beta)}} - \frac{\pi \left[\frac{T^{T(\beta)'}}{T} \right]}{[T^{T(\beta)}]^2}, \quad \frac{\partial ATC\pi}{\partial \beta} = \frac{\pi'_\beta}{T^{T(\beta)}} - \frac{\pi \left[\frac{T^{T(\beta)'}}{\beta} \right]}{[T^{T(\beta)}]^2},$$

$$\frac{\partial^2 ATC\pi}{\partial T^2} = \frac{\pi''_T}{T^{T(\beta)}} - \frac{2\pi'_T \left[\frac{T^{T(\beta)'}}{T} \right] + \pi \left[\frac{T^{T(\beta)''}}{T} \right]}{[T^{T(\beta)}]^2} + \frac{2\pi \left[\frac{T^{T(\beta)'}}{T} \right]^2}{[T^{T(\beta)}]^3},$$

$$\frac{\partial^2 ATC\pi}{\partial \beta^2} = \frac{\pi''_\beta}{T^{T(\beta)}} - \frac{2\pi'_\beta \left[\frac{T^{T(\beta)'}}{\beta} \right] + \pi \left[\frac{T^{T(\beta)''}}{\beta} \right]}{[T^{T(\beta)}]^2} + \frac{2\pi \left[\frac{T^{T(\beta)'}}{\beta} \right]^2}{[T^{T(\beta)}]^3}$$

4.5. Numerical Example

To test out the model and find a possible solution, we use the following exogenous parameters within the objective function to be maximized by finding the second period cycle time (T^*) and preservation level (β^*):

$$Q = 200000, D = 35, \lambda = 0.1, \alpha = 0.2$$

$$P = 30, C_d = 30, C_o = 10, C_h = 0.1, C_p = 0.1$$

5. Solution Procedure

We first specify the range for the two decision variables (T) and (β) that adhere to the assumptions of the model. The objective function is then proven to be strictly concave within the range of the two decision variables. The existence and uniqueness of a solution set (β^*, T^*) is then proven using the strict concavity properties. An algorithm based on the generalized reduced gradient method to find the optimal solution is utilized as previously implemented [20, 25, 26, 37, 39]. Finally, the maximal nature of the solution is verified through the Hessian matrix as approached by previous research [33, 35, 36].

5.1. Range of Decision Variables

The range of (T) can be obtained from the explicit expression of (t_0) (lemma 5). To ensure deterioration kicks-in within the cycle time and not before the order is received, we need:

$$t_0 > 0 \text{ or } \frac{1}{\lambda} \ln \left[\frac{Q(\lambda+\theta)}{[\exp((\lambda+\theta)T)-1]D} \right] > 0 \quad (16)$$

Which can only be observed if:

$$\frac{Q(\lambda+\theta)}{[\exp((\lambda+\theta)T)-1]D} > 1 \text{ or, } \frac{1}{\lambda+\theta} \ln \left[\frac{(\lambda+\theta)Q}{D} + I \right] > T$$

Since:

$$\text{Log}(x) > 0 \text{ when } x > 1 \text{ for all } x \in \mathbb{Q}^+$$

This gives way to the upper bound for our decision variable (T):

$$T \in \{0, \bar{T}\} \text{ where, } \bar{T} = \frac{1}{\lambda+\theta} \ln \left[\frac{(\lambda+\theta)Q}{D} + I \right] \quad (17)$$

For decision variable (β), consider that we need a positive gain of inventory-proportional profit with the use of preservation: $K = \left[\pi_2^\beta \right] - \left[\pi_2^0 \right] > 0$

$$K = [P\lambda - \frac{C_h}{\beta} - C_p\beta - \frac{C_d\alpha}{1+\beta}] - [P\lambda - C_h - C_d\alpha]$$

Since the preservation term (β) is only found within $\left(\pi_2^\beta \right)$, K is thus concave in (β):

$$\frac{\partial K}{\partial \beta} = \frac{\partial \pi_2^\beta}{\partial \beta} > 0 \text{ and } \frac{\partial^2 K}{\partial \beta^2} = \frac{\partial^2 \pi_2^\beta}{\partial \beta^2} < 0$$

The maximum gain of inventory-proportional profit is thus at:

$$\frac{\partial^2 K}{\partial \beta^2} = 6C_p\beta + 2[C_p - C_d\alpha - C_h] = 0 \text{ or,}$$

$$\beta = \frac{[C_d\alpha + C_h - C_p]}{3C_p}$$

Beyond this point, additional investment in preservation is an over-investment since further increase in (β) decrease inventory-proportional profit. Therefore preservation range is determined by inventory dependent cost and the natural deterioration rate (Lemma 6):

$$\beta \in \{0, \bar{\beta}\} \text{ where } \bar{\beta} = \frac{[C_d\alpha + C_h - C_p]}{3C_p} \quad (18)$$

5.2. Concavity Properties

(Theorem 1) The objective function is strictly concave within $T \in \{0, \bar{T}\}$.

Proof: We obtained the range of (T) from Lemma 5 previously:

$$T \in \{0, \bar{T}\} \text{ where,}$$

$$\bar{T} = \frac{1}{\lambda+\theta} \ln \left[\frac{(\lambda+\theta)Q}{D} + I \right] \text{ and } t_0(\bar{T}) = 0$$

Proving the statement below proves this theorem:

$$\delta ATC\pi(0) + (1-\delta) ATC\pi(\bar{T}) <$$

$$ATC\pi[\delta(0) + (1-\delta)(\bar{T})] \text{ for } \delta \in \{0, 1\}$$

Since $[ATC\pi(0) = 0]$, the condition above simplifies into:

$$ATC\pi(\bar{T}) - (\delta) ATC\pi(\bar{T}) < ATC\pi[(1-\delta)(\bar{T})]$$

Consider the case when the value of (δ) is such that:

$$\bar{T} = [(\delta)\bar{T}] + [(1-\delta)\bar{T}] = t_0 + T$$

We know that:

$$ATC\pi[(1-\delta)\bar{T}] > ATC\pi(\bar{T})$$

Since $ATC\pi[(1-\delta)\bar{T}]$ will involve a delay before deterioration (when deterioration starts taking effect with deterioration cost) and have a higher inventory-proportional profit over the total cycle time (\bar{T}) compared to $ATC\pi(\bar{T})$. This proves the above condition and the theorem. \square

(Theorem 2) The objective function is strictly concave within $\beta \in \{0, \bar{\beta}\}$.

Proof: We obtained the range of preservation from Lemma 6 previously:

$$\beta \in \{0, \bar{\beta}\} \text{ where } \bar{\beta} = \frac{[C_d\alpha + C_h - C_p]}{3C_p}$$

Proving Theorem 2 now simplifies into proving strict concavity of the objective function, which requires:

$$ATC\pi[\delta(0) + (1-\delta)(\bar{\beta})] >$$

$$\delta ATC\pi(0) + (1-\delta) ATC\pi(\bar{\beta}) \text{ for } \delta \in \{0, 1\}$$

Or alternatively:

$$ATC\pi[(1-\delta)(\bar{\beta})] > \delta ATC\pi(0) + (1-\delta) ATC\pi(\bar{\beta})$$

We may assume the optimal value of (T^*) since the range of (β) is independent if (T). The condition above can be proven using the set of exogeneous variables in the numerical example to firstly determine ($\bar{\beta}$) before calculating the value of the maximized objective function ($ATC\pi$) for each level of (β). \square

(Theorem 3) There exists a unique optimal solution set for the objective function.

Proof: Both (T^*) and (β^*) lie within their respective closed sets. Since the objective function is also continuous on the bounded sets, there exists at least one maximum point for the objective function. Through Theorem 1 and Theorem 2, we proved that the objective function is strictly concave within the range of our 2 decision variables. Therefore a maximum point obtained by the solution set (β^* , T^*) must be a global maximum since any strictly concave function will have at most one maximal point. \square

5.3. Algorithm

- 1) Set ($\beta_n = C_h$) and ($T_n = I$) with ($n=0$).
- 2) If $\left(\frac{\partial ATC\pi}{\partial \beta} \right)$ and $\left(\frac{\partial ATC\pi}{\partial T} \right)$ are both negative, go to step 8.
- 3) If $\left(\frac{\partial ATC\pi}{\partial \beta} < 0 \right)$, skip to step 5.
- 4) Solve for optimal (β), If $(\Delta ATC\pi > 0.1)$, update (β_n) to the newly solve value.

- 5) If $(\frac{\partial ATC\pi}{\partial T} < 0)$, skip to step 7.
- 6) Solve for optimal (T) , if $(\Delta ATC\pi > 0.1)$, update (T_n) to the newly solved value.
- 7) If $|\beta_n - \beta_{n-1}| + |T_n - T_{n-1}| > 0.001$ go to step 3. If not, go to step 8.
- 8) STOP. The last updated value is an optimal solution set (β^*, T^*) .

We find the optimal cycle time $(T^* = 20.961)$ and optimal level of preservation level $(\beta^* = 0.65754)$ through the Algorithm with an Average Total Cycle Profit $(ATC\pi)$ of (64964.58) .

To establish the maximal nature of our obtained solution,

we verify the principal minor and major of the Hessian matrix evaluated at the solution set (β^*, T^*) . The negative value for the principal minor (concavity for either decision variable) was proven through Theorem 1 and 2. We can check the positive definite sign for the principal major. This reaffirms the maximum point obtained from our solution:

$$|H_2| = \left[\frac{\partial^2 ATC\pi}{\partial \beta^2} \right] \left[\frac{\partial^2 ATC\pi}{\partial T^2} \right] - \left[\frac{\partial^2 ATC\pi}{\partial \beta \partial T} \right]^2 > 0$$

The graphical representation for the objective function against the two decision variable, the level of Preservation (β) and second period cycle time (T) is given below:

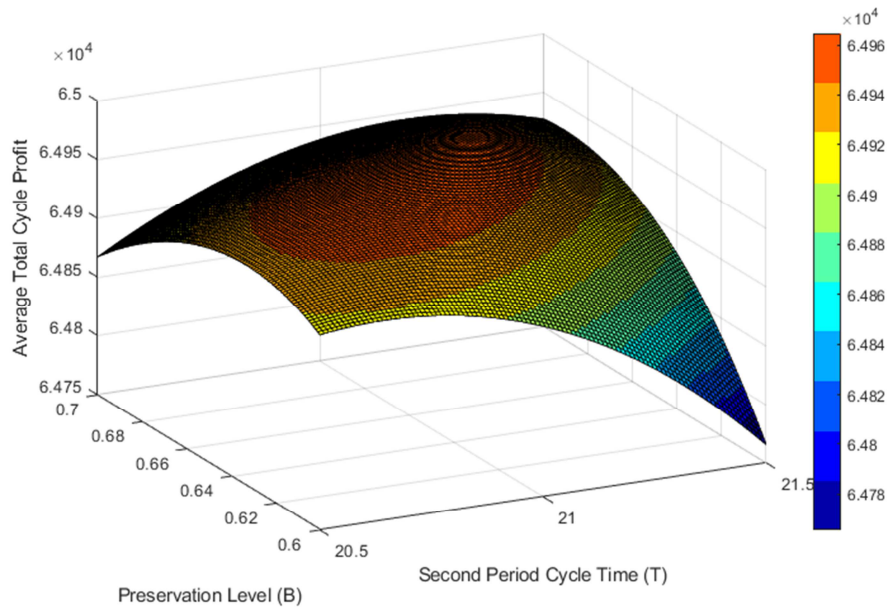


Figure 2. $ATC\pi$ (y-axis) with variation in Total Cycle Time (T) (z-axis) and Preservation Level (β) (x-axis).

5.4. Gains from Using Preservation Technology

While it is clear to see the optimal solutions give the maximum point through the graph, the channels by which preservation increases profit is still uncertain. Since there is evident gains (and possible loss) that could come from the level of preservation, further investigation is needed to actually understand how preservation technology can affect profits. The difference in Average Total Cycle Profit between businesses that uses preservation $(ATC\pi^{(\beta)})$ and a model that does not $(ATC\pi^{(0)})$ can be organized into each of their related exogenous terms. We look into all 5 terms separately:

$$ATC\pi^{(\beta)} - ATC\pi^{(0)} = A + B + C + D + E \quad (19)$$

$$A = [P\lambda] * \left[\frac{I_1^{T(\beta)} + I_2^{T(\beta)}}{T^{T(\beta)}} - \frac{I_1^{T(0)} + I_2^{T(0)}}{T^{T(0)}} \right] \quad (20)$$

For A, the difference between average inventory held for the model with and without preservation determines how much gain it contributes. Through Lemma 3 it is proven that inventory held during the first period increase with higher levels of preservation. As a balance, inventory held during second period decreases through Lemma 4. Nonetheless, notice that total cycle time increases with higher usage of

preservation according to Lemma 2. Therefore over-investment in preservation may eliminate this gain.

$$B = [-C_h] * \left[\frac{I_1^{T(\beta)} + I_2^{T(\beta)}}{\beta T^{T(\beta)}} - \frac{I_1^{T(0)} + I_2^{T(0)}}{T^{T(0)}} \right] \quad (21)$$

For B, the term in the right bracket is similar as before, this time greater inventory held would entail greater total holding cost. Additionally the inventory dependent per unit time holding cost with preservation is discounted proportionally, reducing total holding cost when preservation is used.

$$C = [-C_p\beta] * \left[\frac{I_1^{T(\beta)} + I_2^{T(\beta)}}{T^{T(\beta)}} \right] \quad (22)$$

For C, the preservation cost which is only applicable when preservation is employed increases firstly according to the level of utilization in the effective preservation cost in the left bracket. Secondly the preservation cost is incurred through the average inventory held during the whole cycle.

$$D = [C_d] * \left[\frac{I_2^{T(0)}\alpha}{T^{T(0)}} - \frac{I_2^{T(\beta)}\alpha}{T^{T(\beta)}} \right] \quad (23)$$

Next, term D is the difference in deterioration cost incurred when preservation is utilized. This is where the

majority of cost is saved in deteriorating inventory models with preservation technologies. Through Lemma 4, it can be seen that average inventory held during the second period is smaller with the use of preservation. Additionally, the effective deterioration rate is reduced through Lemma 1.

$$E = [C_0Q] * [\frac{1}{T(0)} - \frac{1}{T(\beta)}] \quad (24)$$

Finally, the last term is the difference in spread of the initial ordering quantity cost based on Lemma 2. Since the length of first period is greater with preservation, the initial ordering cost is spread over a longer period with higher preservation levels.

The gains in $(ATC\pi)$ occur through the alteration of inventory process, which prolongs the delay before deterioration (first period cycle time) and reduces the effective deterioration $\theta < \alpha$ (Preposition 1). By prolonging the delay before preservation, the total costs are spread further as most evidently seen in term E. Meanwhile by reducing the effective deterioration rate, the amount of inventory that is lost to deterioration is reduced as seen in term D. Additionally, the amount of inventory held in the first period is increased while the amount of inventory held in the second period is reduced as proven through Lemmas 3 and 4. As a whole, this shifts the weight of the operation cycle towards the first period where no deterioration cost is incurred.

The gains in $(ATC\pi)$ affected by the changes in inventory-associated costs namely the additional preservation cost $(C_p\beta)$ and the savings of holding cost due to utilization of preservation $(\frac{C_h}{\beta})$ (Preposition 2). We can see this effect clearly within term C where successive increments in preservation will quickly increase through the per unit preservation cost in the left bracket. On the contrary in term B, successive increments in preservation has a positive effect by improving inventory durability that reduces per unit holding cost.

6. Sensitivity Analysis

We conduct sensitivity analysis to verify Prepositions 1 and 2. The tables below show the differences in Average Total Cycle Profit with differing levels of Total Cycle Time and Preservation Level when exogenous costs change.

Consider an increase in deterioration cost from 30 to 34 and subsequently 38. From term D previously, we found in Preposition 1 that it would be beneficial to increase preservation levels higher to reduce the amount of inventory susceptible to deterioration in the second period. The more expensive deterioration cost, the more gains that could come from increasing preservation levels as seen in tables 2 and 3:

Table 3. $ATC\pi$ values for $C_d = 34$.

β/T	19	20	20.961	21	22
0.55	64584	64757	64577	64527	63755
0.6	64624	64878	64812	64780	64211
0.65	64566	64889	64923	64907	64518
0.7	64464	64830	64927	64921	64649
0.75	64305	64709	64865	64868	64704

Table 4. $ATC\pi$ values for $C_d = 38$.

β/T	19	20	20.961	21	22
0.55	64234	64299	63997	63930	62976
0.6	64304	64461	64286	64239	63509
0.65	64275	64511	64448	64418	63888
0.7	64192	64479	64488	64469	64068
0.75	64053	64385	64460	64452	64172

We now consider changes in Preservation Cost from 0.1 to 0.3 and subsequently 0.5. From term C of section 5.8, it is discussed through Preposition 2 that it will be more expensive to employ higher preservation levels, making it less beneficial to have high levels of preservation. Observing tables 4 and 5 below confirm predictions from Preposition 2:

Table 5. $ATC\pi$ values for $C_p = 34$.

β/T	19	20	20.961	21	22
0.55	60706	60901	60770	60729	60061
0.6	60378	60638	60604	60578	60089
0.65	59919	60234	60283	60271	59937
0.7	59503	59851	59952	59948	59715
0.75	58997	59372	59520	59524	59383

Table 6. $ATC\pi$ values for $C_p = 38$.

β/T	19	20	20.961	21	22
0.55	56477	56587	56384	56333	55589
0.6	55810	55982	55871	55835	55263
0.65	54980	55201	55169	55147	54726
0.7	54271	54521	54538	54524	54199
0.75	53436	53710	53771	53764	53528

We graph the data to show optimal level of preservations does follow the notions of prepositions 1 and 2. Increasing deterioration costs entails greater optimal preservation level while increasing preservation costs will require lower levels of optimal preservation:

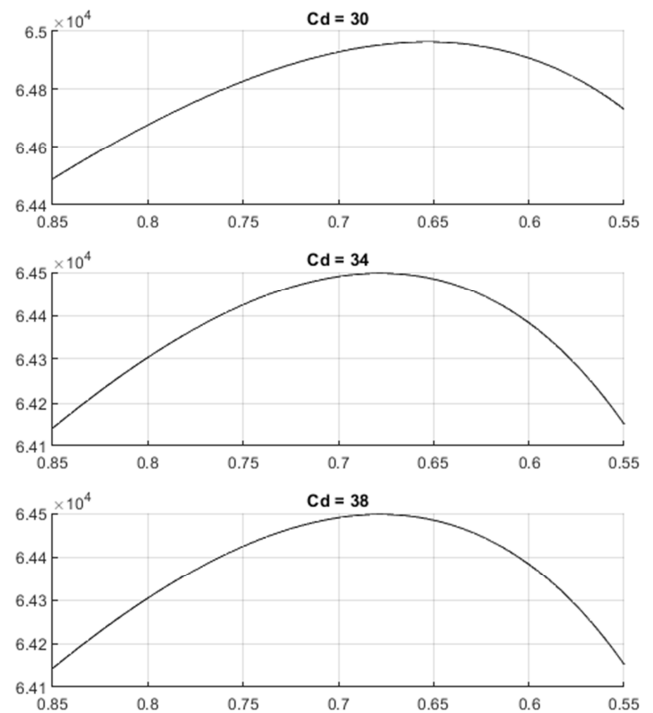


Figure 3. Optimal Preservation Levels (β) with increase in Deterioration Cost (C_d).

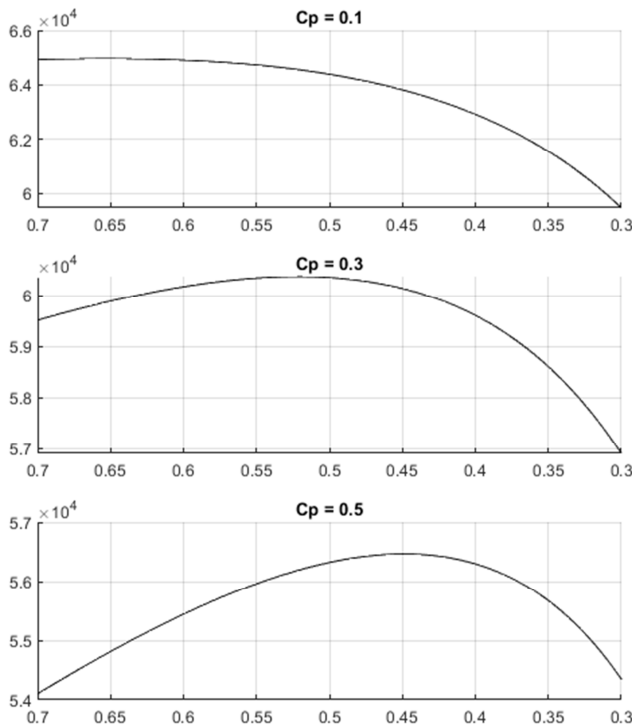


Figure 4. Optimal Preservation Levels (β) with increase in Preservation Cost (C_p).

Gains in employing additional levels of preservation would depend on the returns to scale of the preservation technology towards inventory related cost and effective deterioration rate. (Preposition 3)

Consider the additional dimension of inventory-associated cost where there are differing returns to scale. Increasing levels of preservation could require increasing increments to the preservation cost. Now increase in preservation levels increases preservation cost at an increasing rate. This would limit the gain from preservation, lowering the optimal level of preservation. This is more clearly seen where term C changes into:

$$C = [-C_p \beta^n] \left[\frac{I_1^{T(\beta)} + I_2^{T(\beta)}}{T^{T(\beta)}} \right] \text{ where } (n > 1) \quad (25)$$

Equally interesting would be decreasing scale to holding cost savings. Now there would be less difference in total holding cost when preservation is employed since higher level of preservation would not continuously reduce holding cost. Similar as before, this would lower the optimal level of preservation. This can be seen where term B changes into:

$$B = [-C_h] \left[\frac{I_1^{T(\beta)} + I_2^{T(\beta)}}{(\beta^m) T^{T(\beta)}} - \frac{I_1^{T(0)} + I_2^{T(0)}}{T^{T(0)}} \right] \quad (26)$$

where, ($m < 1$)

The final consideration is varying effectiveness for preservation in reducing effective deterioration as seen below:

$$\theta(k) = \frac{\alpha}{1 + (\beta^k)}, I_2(t) = \frac{D}{\lambda + \theta k} [\exp((\lambda + \theta(k))(T-t)) - 1] \quad (27)$$

This would alter the initial condition that implicitly

determines first period cycle time (t_0) and total cycle time. The value of (k) would also determine the extent of delay that would be incurred via the preservation investment. Values of ($k > 1$) would signal increasing effectiveness of preservation investment in reducing effective deterioration. The same amount of decrease in effective deterioration rate would require a lower level of preservation as opposed to an ineffective preservation situation with ($k < 1$) values. Ultimately this will affect total inventory held during both periods, a major component in the objective function and affecting all 5 terms relating to the gains from preservation.

7. Future Research

Possible extensions and improvements for this study may consider further detailed effects of preservation technologies in its effectiveness and the salvage value for deteriorated items that have been preserved.

As implied through Preposition 3, differing returns to scale and increments in preservation costs could further explain until which exact point preservation continues to give benefit to the business. The benefits to employing preservation technology may then be attributed to each specific cost or revenue component rather than just the overall profit. Such an approach has not been taken since previous papers consider linearly increasing preservation costs even with diminishing returns to scale for preservation.

Finally, there is no salvage value consideration and discussion. Since inventories are well kept and preserved during the whole cycle, deteriorated items could be sold off at a discount once they deteriorate [22, 36, 40]. There could be cost recuperated from this avenue that should be explored.

8. Conclusion

In this paper, it is demonstrated that the average total cycle profit could be maximized by choosing the optimal level of preservation (β^*) and second period cycle time (T^*) in a non-instantaneous deteriorating inventory setting. We discover two channels by which preservation increases profit, namely by altering the inventory process and the changes in inventory costs affected by the level of preservation utilized.

Through the alteration of inventory process, we find that a higher level of preservation prolongs the delay before deterioration and increases the inventory held during the first period. This essentially shifts weight and majority of the operation cycle to the higher profitable first period in line with Preposition 1. The derivation on how the level of preservation determines the time when inventory process changes is explicitly demonstrated in this research, improving the current limited literature on this issue.

Additionally, it is found with Preposition 2 that businesses stand to gain (or lose) depending on the inventory dependent costs incurred during the cycle. Favorably, preservation cost is now incurred more accurately according to the level of inventory held. Holding costs during the cycle are also reduced owing to more durable inventory requiring less

delicate care. While it is beneficial to employ sufficiently high levels of preservation, successively increasing preservation levels might require increasing increments to the preservation cost as suggested through Proposition 3.

Industries and businesses with easily-perishable high value products like caviar and radioactive materials would greatly benefit from the findings of this research. There are great savings that would arise from the reduced deterioration cost and inventory preserved to satisfy demand when the appropriate operation cycle time and level of preservation are

chosen. Future research in this area could consider varying preservation technologies effectiveness that have diminishing or increasing increments towards inventory related costs.

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Appendix: Order Conditions

A) Order conditions for per inventory proportional profit in period 1 and 2:

$$\begin{aligned}\pi_1(\beta) &= [P\lambda - \frac{C_h}{\beta} - C_p\beta] \quad \pi_2(\beta) = [P\lambda - \frac{C_h}{\beta} - C_p\beta - \frac{C_d\alpha}{1+\beta}] \\ \pi_{1\beta}' &= [\frac{C_h}{\beta^2} - C_p] \quad \pi_{2\beta}' = [\frac{C_h}{\beta^2} - C_p - \frac{C_d\alpha}{(1+\beta)^2}] \\ \pi_{1\beta}'' &= [-\frac{2C_h}{\beta^3} - C_p] \quad \pi_{2\beta}'' = -2 [\frac{C_h}{\beta^3} + \frac{C_d\alpha}{(1+\beta)^3}]\end{aligned}$$

B) Order conditions for Inventory held in period 1:

$$\begin{aligned}I_1^{T(\beta)}(\beta, T) &= \int_0^{t_0} I_1(t) dt = \frac{Q}{\lambda} [I - \frac{1}{\exp(\lambda t_0)}] \\ I_{1T}^{T(\beta)'} &= \frac{Q(\lambda+\theta)\exp((\lambda+\theta)T)}{[\lambda \exp(\lambda t_0)][1 - \exp((\lambda+\theta)T)]} \\ I_{1T}^{T(\beta)''} &= \frac{Q(\lambda+\theta)\exp((\lambda+\theta)T)}{[\lambda \exp(\lambda t_0)][\exp((\lambda+\theta)T) - 1]} \\ I_{1\beta}^{T(\beta)'} &= \frac{Q\alpha\lambda}{[(1+\beta)\lambda \exp(\lambda t_0)]^2} \left\{ \frac{T \exp((\lambda+\theta)T)}{[\exp((\lambda+\theta)T) - 1]} - (\lambda + \theta) \right\} \\ I_{1\beta}^{T(\beta)''} &= \frac{Q\alpha^3\lambda}{[(1+\beta)^4[\lambda \exp(\lambda t_0)]]^2} \left\{ \frac{T^2 \exp((\lambda+\theta)T)}{[\exp((\lambda+\theta)T) - 1]} + 1 \right\} \\ &+ \left\{ \frac{Q\alpha\lambda^3 \exp(\lambda t_0)}{[(1+\beta)^4[\lambda \exp(\lambda t_0)]]^4} \left[\frac{T \exp((\lambda+\theta)T)}{[\exp((\lambda+\theta)T) - 1]} - (\lambda + \theta) \right] \right\} \\ &\left\{ [(\lambda + \theta) - \frac{T \exp((\lambda+\theta)T)}{[\exp((\lambda+\theta)T) - 1]}] - [\lambda \exp((\lambda + \theta)T)] \right\}\end{aligned}$$

C) Order conditions for Inventory held in period 2:

$$\begin{aligned}I_2^{T(\beta)}(\beta, T) &= \int_0^T I_2(t) dt = \frac{D}{\lambda+\theta} \left[\frac{\exp((\lambda+\theta)T) - 1}{\lambda+\theta} - T \right] \\ I_{2T}^{T(\beta)'} &= \frac{D[\exp((\lambda+\theta)T) - 1]}{\lambda+\theta}, \quad I_{2T}^{T(\beta)''} = D[\exp((\lambda + \theta)T)] \\ I_{2\beta}^{T(\beta)'} &= \frac{\alpha D}{(\lambda+\theta)^2(1+\beta)^2} \left\{ (\lambda + \theta)\exp((\lambda + \theta)T) [2 - ((\lambda + \theta)T)] - 2(\lambda + \theta) - T \right\} \\ I_{2\beta}^{T(\beta)''} &= \left\{ \frac{2[\alpha D][\lambda+\theta][\alpha - (\lambda+\theta)(1+\beta)]}{(\lambda+\theta)^4(1+\beta)^4} \right\} \left\{ (\lambda + \theta)\exp((\lambda + \theta)T) [2 - ((\lambda + \theta)T)] - 2(\lambda + \theta) - T \right\} + \\ &\left\{ \frac{\alpha D}{(\lambda+\theta)^2(1+\beta)^4} \right\} \left\{ 2(\lambda + \theta)\alpha + [T(\lambda + \theta)\exp((\lambda + \theta)T)][T(\lambda + \theta) + 2] - 2\alpha \exp((\lambda + \theta)T)[(\lambda + \theta) + 1] \right\}\end{aligned}$$

D) Order conditions for end of period 1:

$$\begin{aligned}t_0 &= \frac{1}{\lambda} \ln \left[\frac{Q(\lambda+\theta)}{[\exp((\lambda+\theta)T) - 1]D} \right] \\ t_{0\beta}' &= \frac{\alpha}{\lambda(\lambda+\theta)(1+\beta)^2} \left[\frac{T \exp((\lambda+\theta)T)(\lambda+\theta)}{[\exp((\lambda+\theta)T) - 1]} - 1 \right]\end{aligned}$$

$$t_{0T}' = \frac{\exp((\lambda+\theta)T)(\lambda+\theta)}{\lambda [1 - \exp((\lambda+\theta)T)]}$$

$$t_{0\beta}'' = \frac{\alpha^2}{\lambda(\lambda+\theta)(1+\beta)^4} \left[\frac{T^2 \exp((\lambda+\theta)T)}{[\exp((\lambda+\theta)T)-1]} - \frac{1}{(\lambda+\theta)^2} \right] + \frac{\alpha[\alpha-2(\lambda+\theta)]}{\lambda(\lambda+\theta)^2(1+\beta)^3} \left[\frac{T^2 \exp((\lambda+\theta)T)}{[\exp((\lambda+\theta)T)-1]} - \frac{1}{(\lambda+\theta)} \right]$$

$$t_{0T}'' = \frac{\alpha[1 - \exp((\lambda+\theta)T)]^2 \exp((\lambda+\theta)T)}{(1+\beta)^2} \{ ((\lambda+\theta)T) - 1 - (\lambda+\theta)T \}$$

E) Order conditions for total cycle time:

$$T^{T(\beta)} = t_0 + T$$

$$T_T^{T(\beta)'} = t_{0T}' + 1 \quad T_T^{T(\beta)''} = t_{0T}''$$

$$T_\beta^{T(\beta)'} = t_{0\beta}' \quad T_\beta^{T(\beta)''} = t_{0\beta}''$$

F) Order conditions for total profit:

$$\pi = [PDT] - [C_oQ] + [\pi_1 I_1^{T(\beta)}] + [\pi_2 I_2^{T(\beta)}]$$

$$\pi_T' = [PD] + [\pi_1 I_{1T}^{T(\beta)'}] + [\pi_2 I_{2T}^{T(\beta)'}]$$

$$\pi_T'' = [\pi_1 I_{1T}^{T(\beta)''}] + [\pi_2 I_{2T}^{T(\beta)''}]$$

$$\text{Since } \pi_{1T}' = 0, \pi_{1T}'' = 0, \pi_{2T}' = 0, \pi_{2T}'' = 0$$

$$\pi = [PDT] - [C_oQ] + [\pi_1 I_1^{T(\beta)}] + [\pi_2 I_2^{T(\beta)}]$$

$$\pi_\beta' = (\pi_{1\beta}' I_1^{T(\beta)} + \pi_1 I_{1\beta}^{T(\beta)'}) + (\pi_{2\beta}' I_2^{T(\beta)} + \pi_2 I_{2\beta}^{T(\beta)'})$$

$$\pi_\beta'' = [(\pi_{1\beta}'' I_1^{T(\beta)} + \pi_{1\beta}' I_{1\beta}^{T(\beta)'}) + (\pi_{1\beta}' I_{1\beta}^{T(\beta)''} + \pi_1 I_{1\beta}^{T(\beta)''})] + [(\pi_{2\beta}'' I_2^{T(\beta)} + \pi_{2\beta}' I_{2\beta}^{T(\beta)'}) + (\pi_{2\beta}' I_{2\beta}^{T(\beta)''} + \pi_2 I_{2\beta}^{T(\beta)''})]$$

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